

# MOMENT SERIES AND R-TRANSFORM OF THE GENERATING OPERATOR OF $L(F_N)$

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**ABSTRACT.** In this paper, we will consider the free probability theory of free group factor  $L(F_N)$ , where  $F_k$  is the free group with  $k$ -generators. We compute the moment series and the R-transform of the generating operator,  $T = g_1 + \dots + g_N + g_1^{-1} + \dots + g_N^{-1}$ .

Voiculescu developed Free Probability Theory. Here, the classical concept of Independence in Probability theory is replaced by a noncommutative analogue called Freeness (See [9]). There are two approaches to study Free Probability Theory. One of them is the original analytic approach of Voiculescu and the other one is the combinatorial approach of Speicher and Nica (See [1], [2] and [3]). Speicher defined the free cumulants which are the main objects in Combinatorial approach of Free Probability Theory. And he developed free probability theory by using Combinatorics and Lattice theory on collections of noncrossing partitions (See [3]). Also, Speicher considered the operator-valued free probability theory, which is also defined and observed analytically by Voiculescu, when  $\mathbb{C}$  is replaced to an arbitrary algebra  $B$  (See [1]). Nica defined R-transforms of several random variables (See [2]). He defined these R-transforms as multivariable formal series in noncommutative several indeterminates. To observe the R-transform, the Möbius Inversion under the embedding of lattices plays a key role (See [1],[3],[5],[12]). In [12], we observed the amalgamated R-transform calculus. Actually, amalgamated R-transforms are defined originally by Voiculescu (See [10]) and are characterized combinatorially by Speicher (See [1]). In [12], we defined amalgamated R-transforms slightly differently from those defined in [1] and [10]. We defined them as  $B$ -formal series and tried to characterize, like in [2] and [3]. In [13], we observed the compatibility of a noncommutative probability space and an amalgamated noncommutative probability space over an unital algebra. In [14], we found the amalgamated moment series, the amalgamated R-transform and the scalar-valued moment series and the scalar-valued R-transform of the generating operator of  $\mathbb{C}[F_2] *_{\mathbb{C}[F_1]} \mathbb{C}[F_2]$ ,

$$a + b + a^{-1} + b^{-1} + c + d + c^{-1} + d^{-1}$$

where  $\langle a, b \rangle = F_2 = \langle c, d \rangle$ . The moment series and R-transforms (operator-valued or scalar-valued) of the above generating operator is determined by the recurrence relations. In this paper, by using one of the recurrence relation found

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in [14], we will consider the moment series and the R-transform of the generating operator of  $L(F_N)$ ,

$$G = g_1 + \dots + g_N + g_1^{-1} + \dots + g_N^{-1} \in L(F_N),$$

where  $F_N = \langle g_1, \dots, g_N \rangle$ , for  $N \in \mathbb{N}$ .

## 1. PRELIMINARIES

Let  $A$  be a von Neumann algebra and let  $\tau : A \rightarrow \mathbb{C}$  be the normalized faithful trace. Then we call the algebraic pair  $(A, \tau)$ , the  $W^*$ -probability space and we call elements in  $(A, \tau)$ , random variables. Define the collection  $\Theta_s$ , consists of all formal series without the constant terms in noncommutative indeterminants  $z_1, \dots, z_s$  ( $s \in \mathbb{N}$ ). Then we can regard the moment series of random variables and R-transforms of random variables as elements of  $\Theta_s$ . In fact, for any element  $f \in \Theta_s$ , there exists (some) noncommutative probability space  $(A, \tau)$  and random variables  $x_1, \dots, x_s \in (A, \tau)$  such that

$$f(z_1, \dots, z_s) = R_{x_1, \dots, x_s}(z_1, \dots, z_s),$$

where  $R_{x_1, \dots, x_s}$  is the R-transform of  $x_1, \dots, x_s$ , by Nica and Speicher.

**Definition 1.1.** Let  $(A, \tau)$  be a  $W^*$ -probability space and let  $a_1, \dots, a_s \in (A, \tau)$  be random variables ( $s \in \mathbb{N}$ ). Define the moment series of  $a_1, \dots, a_s$  by

$$M_{a_1, \dots, a_s}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} \tau(a_{i_1} \dots a_{i_n}) z_{i_1} \dots z_{i_n} \in \Theta_s.$$

Define the R-transform of  $a_1, \dots, a_s$  by

$$R_{a_1, \dots, a_s}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} k_n(a_{i_1}, \dots, a_{i_n}) z_{i_1} \dots z_{i_n} \in \Theta_s.$$

And we say that the  $(i_1, \dots, i_n)$ -th coefficient of  $M_{a_1, \dots, a_s}$  and that of  $R_{a_1, \dots, a_s}$  are the joint moment of  $a_1, \dots, a_s$  and the joint cumulant of  $a_1, \dots, a_s$ , respectively.

By Speicher and Nica, we have that

**Proposition 1.1.** (See [1], [2] and [3]) Let  $(A, \tau)$  be a noncommutative probability space and let  $a_1, \dots, a_s \in (A, \tau)$  be random variables. Then

$$k_n(a_{i_1}, \dots, a_{i_n}) = \sum_{\pi \in NC(n)} \tau_\pi(a_{i_1}, \dots, a_{i_n}) \mu(\pi, 1_n),$$

where  $NC(n)$  is the collection of all noncrossing partitions and  $\mu$  is the Möbius function in the incidence algebra and where  $\tau_\pi$  is the partition-dependent moment in the sense of Nica and Speicher (See [2] and [3]), for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $n \in \mathbb{N}$ . Equivalently,

$$\tau(a_{i_1} \dots a_{i_n}) = \sum_{\pi \in NC(n)} k_\pi(a_{i_1}, \dots, a_{i_n}),$$

where  $k_\pi$  is the partition-dependent cumulant in the sense of Nica and Speicher, for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $n \in \mathbb{N}$ .  $\square$

The above combinatorial moment-cumulant relation is so-called the Möbius inversion. The R-transforms play a key role to study the freeness and the R-transform calculus is well-known (See [2] and [3]).

Let  $H$  be a group and let  $L(H)$  be a group von Neumann algebra i.e

$$L(H) = \overline{\mathbb{C}[H]}^w.$$

Precisely, we can regard  $L(H)$  as a weak-closure of group algebra generated by  $H$  and hence

$$L(H) = \overline{\left\{ \sum_{g \in H} t_g g : g \in H \right\}}^w.$$

It is well known that  $L(H)$  is a factor if and only if the given group  $H$  is icc. Since our object  $F_N$  is icc, the von Neumann group algebra  $L(F_N)$  is a factor. Now, define a trace  $\tau : L(H) \rightarrow \mathbb{C}$  by

$$\tau \left( \sum_{g \in H} t_g g \right) = t_{e_H}, \text{ for all } \sum_{g \in H} t_g g,$$

where  $e_H$  is the identity of the group  $H$ . Then  $(L(H), \tau)$  is the (group)  $W^*$ -probability space. Notice that  $L(F_N)$  is a  $\text{II}_1$ -factor under this trace  $\tau$ . Assume that the group  $H$  has its generators  $\{g_j : j \in I\}$ . We say that the operator

$$G = \sum_{j \in I} g_j + \sum_{j \in I} g_j^{-1},$$

the generating operator of  $L(H)$ .

Rest of this paper, we will consider the moment series and the R-transform of the generating operator  $G$  of  $L(F_N)$ .

## 2. THE MOMENT SERIES OF THE GENERATING OPERATOR $G \in L(F_N)$

In this chapter, we will consider free group  $\Pi_1$ -factor,  $A \stackrel{\text{denote}}{=} L(F_N)$ , where  $F_k$  is a free group with  $k$ -generators ( $k \in \mathbb{N}$ ). i.e

$$A = \left\{ \sum_{g \in F_N} t_g g : t_g \in \mathbb{C} \right\}.$$

Recall that there is the canonical trace  $\tau : A \rightarrow \mathbb{C}$  defined by

$$\tau \left( \sum_{g \in F_N} t_g g \right) = t_e,$$

where  $e \in F_N$  is the identity of  $F_N$  and hence  $e \in L(F_N)$  is the unity  $1_{L(F_N)}$ . The algebraic pair  $(L(F_N), \tau)$  is a  $W^*$ -probability space. Let  $G$  be the generating operator of  $L(F_N)$ . i.e

$$G = g_1 + \dots + g_N + g_1^{-1} + \dots + g_N^{-1},$$

where  $F_N = \langle g_1, \dots, g_N \rangle$ . It is known that if we denote  $X_n = \sum_{|w|=n} w \in A$ , for all  $n \in \mathbb{N}$ , then

$$(1.1) \quad X_1 X_1 = X_2 + 2N \cdot e \quad (n = 1)$$

and

$$(1.2) \quad X_1 X_n = X_{n+1} + (2N - 1)X_{n-1} \quad (n \geq 2)$$

(See [36]).

In our case, we can regard our generating operator  $G$  as  $X_1$  in  $A$ .

By using the relation (1.1) and (1.2), we can express  $G^n$  in terms of  $X_k$ 's ; For example,  $G = X_1$ ,

$$G^2 = X_1 X_1 = X_2 + 2N \cdot e,$$

$$\begin{aligned} G^3 &= X_1 \cdot X_1^2 = X_1 (X_2 + (2N)e) = X_1 X_2 + (2N)X_1 \\ &= X_3 + (2N - 1)X_1 + (2N)X_1 = X_3 + ((2N - 1) + 2N)X_1, \end{aligned}$$

continuing

$$\begin{aligned}
G^4 &= X_4 + ((2N-1) + (2N-1) + 2N) X_2 + (2N) ((2N-1) + (2N)) e, \\
G^5 &= X_5 + ((2N-1) + (2N-1) + (2N-1) + 2N) X_3 \\
&\quad + ((2N-1) ((2N-1) + (2N-1) + (2N)) + (2N) ((2N-1) + (2N))) X_1, \\
G^6 &= X_6 + ((2N-1) + (2N-1) + (2N-1) + (2N-1) + 2N) X_4 \\
&\quad + \{(2N-1) ((2N-1) + (2N-1) + (2N-1) + (2N)) \\
&\quad \quad + (2N-1) ((2N-1) + (2N-1) + (2N)) \\
&\quad \quad + (2N-1) ((2N-1) + (2N))\} X_2 \\
&\quad + (2N) ((2N-1) ((2N-1) + (2N-1) + (2N))) + (2N) ((2N-1) + (2N)) e, \\
&\text{etc.}
\end{aligned}$$

So, we can find a recurrence relation to get  $G^n$  ( $n \in \mathbb{N}$ ) with respect to  $X_k$ 's ( $k \leq n$ ). Inductively, we have that  $G^{2k-1}$  and  $G^{2k}$  have their representations in terms of  $X_j$ 's as follows ;

$$G^{2k-1} = X_1^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1} X_{2k-3} + q_{2k-5}^{2k-1} X_{2k-5} + \dots + q_3^{2k-1} X_3 + q_1^{2k-1} X_1$$

and

$$G^{2k} = X_1^{2k} = X_{2k} + p_{2k-2}^{2k} X_{2k-2} + p_{2k-4}^{2k} X_{2k-4} + \dots + p_2^{2k} X_2 + p_0^{2k} e,$$

where  $k \geq 2$ . Also, we have the following recurrence relation ;

**Proposition 2.1.** *Let's fix  $k \in \mathbb{N} \setminus \{1\}$ . Let  $q_i^{2k-1}$  and  $p_j^{2k}$  ( $i = 1, 3, 5, \dots, 2k-1, \dots$  and  $j = 0, 2, 4, \dots, 2k, \dots$ ) be given as before. If  $p_0^2 = 2N$  and  $q_1^3 = (2N-1) + (2N)^2$ , then we have the following recurrence relations ;*

(1) Let

$$G^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1} X_{2k-3} + \dots + q_3^{2k-1} X_3 + q_1^{2k-1} X_1.$$

Then

$$\begin{aligned}
G^{2k} &= X_{2k} + ((2N-1) + q_{2k-3}^{2k-1}) X_{2k-2} + ((2N-1) q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}) X_{2k-4} \\
&\quad + ((2N-1) q_{2k-5}^{2k-1} + q_{2k-7}^{2k-1}) X_{2k-6} + \\
&\quad + \dots + ((2N-1) q_3^{2k-1} + q_1^{2k-1}) X_2 + (2N) q_1^{2k-1} e.
\end{aligned}$$

i.e,

$$\begin{aligned}
p_{2k-2}^{2k} &= (2N-1) + q_{2k-3}^{2k-1}, \\
p_{2k-4}^{2k} &= (2N-1) q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}, \\
&\dots\dots\dots, \\
p_2^{2k} &= (2N-1) q_3^{2k-1} + q_1^{2k-1} \\
&\text{and} \\
p_0^{2k} &= (2N) q_1^{2k-1}.
\end{aligned}$$

(2) Let

$$G^{2k} = X_{2k} + p_{2k-2}^{2k} X_{2k-2} + \dots + p_2^{2k} X_2 + p_0^{2k} e.$$

Then

$$G^{2k+1} = X_{2k+1} + ((2N-1) + p_{2k-2}^{2k}) X_{2k-1} + ((2N-1)p_{2k-2}^{2k} + p_{2k-4}^{2k}) X_{2k-3} \\ + ((2N-1)p_{2k-4}^{2k} + p_{2k-6}^{2k}) X_{2k-5} + \\ + \dots + ((2N-1)p_4^{2k} + p_2^{2k}) X_3 + ((2N-1)p_2^{2k} + p_0^{2k}) X_1.$$

i.e.,

$$q_{2k-1}^{2k+1} = (2N-1) + p_{2k-2}^{2k}, \\ q_{2k-3}^{2k+1} = (2N-1)p_{2k-2}^{2k} + p_{2k-4}^{2k},$$

.....,

$$q_3^{2k+1} = (2N-1)p_4^{2k} + p_2^{2k}$$

and

$$q_1^{2k+1} = (2N-1)p_2^{2k} + p_0^{2k}.$$

□

**Example 2.1.** Suppose that  $N = 2$ . and let  $p_0^2 = 4$  and  $q_1^3 = 3 + p_0^2 = 3 + 4 = 7$ . Put

$$G^8 = X_8 + p_6^8 X_6 + p_4^8 X_4 + p_2^8 X_2 + p_0^8 e.$$

Then, by the previous proposition, we have that

$$p_6^8 = 3 + q_5^7, \quad p_4^8 = 3q_5^7 + q_3^7, \quad p_2^8 = 3q_3^7 + q_1^7 \quad \text{and} \quad p_0^8 = 4q_1^7.$$

Similarly, by the previous proposition,

$$q_5^7 = 3 + p_4^6, \quad q_3^7 = 3p_4^6 + p_2^6 \quad \text{and} \quad q_1^7 = 3p_2^6 + p_0^6, \\ p_4^6 = 3 + q_3^5, \quad p_2^6 = 3q_3^5 + q_1^5 \quad \text{and} \quad p_0^6 = 4q_1^5,$$

$$q_3^5 = 3 + p_2^4 \quad \text{and} \quad q_1^5 = 3p_2^4 + p_0^4,$$

$$p_2^4 = 3 + q_1^3 \quad \text{and} \quad p_0^4 = 4q_1^3,$$

and

$$q_1^3 = 3 + p_0^2 = 7.$$

Therefore, combining all information,

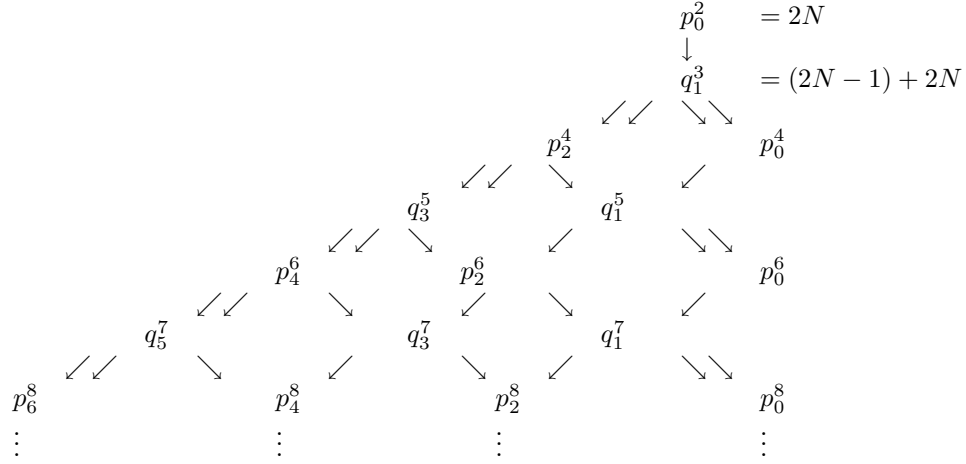
$$G^8 = X_8 + 22 X_6 + 202 X_4 + 744 X_2 + 1316 e.$$

We have the following diagram with arrows which mean that

$$\begin{array}{lcl} \swarrow \swarrow & : & (2N-1) + [\text{former term}] \\ \searrow & : & (2N-1) \cdot [\text{former term}] \\ \swarrow & : & \cdot + [\text{former term}] \end{array}$$

and

$$\searrow \searrow : (2N) \cdot [\text{former term}].$$



Recall that Nica and Speicher defined the even random variable in a  $*$ -probability space. Let  $(B, \tau_0)$  be a  $*$ -probability space, where  $\tau_0 : B \rightarrow \mathbb{C}$  is a linear functional satisfying that  $\tau_0(b^*) = \overline{\tau_0(b)}$ , for all  $b \in B$ , and let  $b \in (B, \tau_0)$  be a random variable. We say that the random variable  $b \in (B, \tau_0)$  is even if it is self-adjoint and it satisfies the following moment relation ;

$$\tau_0(b^n) = 0, \text{ whenever } n \text{ is odd.}$$

In [12], we observed the amalgamated evenness and we showed that  $b \in (B, \tau_0)$  is even if and only if

$$k_n(b, \dots, b) = 0, \text{ whenever } n \text{ is odd.}$$

By the previous observation, we can get that

**Theorem 2.2.** *Let  $G \in (A, \tau)$  be the generating operator. Then the moment series of  $G$  is*

$$\tau(G^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ p_0^n & \text{if } n \text{ is even,} \end{cases}$$

for all  $n \in \mathbb{N}$ .

*Proof.* Assume that  $n$  is odd. Then

$$G^n = X_n + q_{n-2}^n X_{n-2} + \dots + q_3^n X_3 + q_1^n X_1.$$

So,  $G^n$  does not contain the  $e$ -terms. Therefore,

$$\tau(G^n) = \tau(X_n + q_{n-2}^n X_{n-2} + \dots + q_3^n X_3 + q_1^n X_1) = 0.$$

Assume that  $n$  is even. Then

$$G^n = X_n + p_{n-2}^n X_{n-2} + \dots + p_2^n X_2 + p_0^n e.$$

So, we have that

$$\tau(G^n) = \tau(X_n + p_{n-2}^n X_{n-2} + \dots + p_2^n X_2 + p_0^n e) = p_0^n.$$

■

Remark that the  $n$ -th moments of the generating operator in  $(A, \tau)$  is totally depending on the recurrence relation.

**Corollary 2.3.** *Let  $G \in (A, \tau)$  be the generating operator. Then  $G$  is even. □*

**Corollary 2.4.** *Let  $G \in (A, \tau)$  be the generating operator. Then*

$$M_G(z) = \sum_{n=1}^{\infty} p_0^{2n} z^{2n} \in \Theta_1.$$

*Proof.* Since all odd moments of  $G$  vanish,  $\text{coef}_n(M_G) = 0$ , for all odd integer  $n$ . By the previous theorem, we can get the result. ■

### 3. THE R-TRANSFORM OF THE GENERATING OPERATOR OF $L(F_N)$

In this chapter, we will compute the R-transform of the generating operator  $G$  in  $(A, \tau) \equiv (L(F_N), \tau)$ . We can get the R-transform by using the Möbius inversion, which we considered in Chapter 1. Notice that, by the evenness of  $G$ , we have that all odd cumulants of  $G$  vanish. i.e,



$$k_n \left( \underbrace{G, \dots, G}_{n\text{-times}} \right) = 0, \text{ whenever } n \text{ is odd.}$$

This shows that the nonvanishing  $n$ -th coefficients of the R-transform of  $G$ ,  $R_G$ , are all even coefficients.

**Theorem 3.1.** *Let  $G \in (A, \tau)$  be the generating operator. Then  $R_G(z) = \sum_{n=1}^{\infty} \alpha_{2n} z^{2n}$ , in  $\Theta_1$ , with*

$$\alpha_{2n} = \sum_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2n} \sum_{\pi \in NC_{l_1, \dots, l_p}(2n)} \left( p_0^{l_1} \dots p_0^{l_p} \right) \mu(\pi, 1_{2n}),$$

for all  $n \in \mathbb{N}$ , where

$$NC_{l_1, \dots, l_p}(2n) = \{ \pi \in NC(2n) : V \in \pi \Leftrightarrow |V| = l_j, j = 1, \dots, p \}.$$

*Proof.* By the evenness of  $G$ , all odd cumulants vanish (See [12]). Fix  $n \in \mathbb{N}$ , an even number. Then

$$\begin{aligned} k_n \left( \underbrace{G, \dots, G}_{n\text{-times}} \right) &= \sum_{\pi \in NC(n)} \tau_{\pi}(G, \dots, G) \mu(\pi, 1_n) \\ &= \sum_{\pi \in NC^{(even)}(n)} \tau_{\pi}(G, \dots, G) \mu(\pi, 1_n) \end{aligned}$$

where  $NC^{(even)}(n) = \{ \pi \in NC(n) : \pi \text{ does not contain odd blocks} \}$ , by [12]

$$\begin{aligned} &= \sum_{\pi \in NC^{(even)}(n)} \left( \prod_{V \in \pi} \tau(G^{|V|}) \right) \mu(\pi, 1_n) \\ (2.1) \quad &= \sum_{\pi \in NC^{(even)}(n)} \left( \prod_{V \in \pi} p_0^{|V|} \right) \mu(\pi, 1_n). \end{aligned}$$

By [14], we have that

$$NC^{(even)}(n) = \bigsqcup_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = n} NC_{l_1, \dots, l_p}(n)$$

where  $\bigsqcup$  is the disjoint union and

$$NC_{l_1, \dots, l_p}(n) = \{\pi \in NC^{(even)}(n) : V \in \pi \Leftrightarrow |V| = l_j, j = 1, \dots, p\}.$$

(For instance,  $NC^{(even)}(6) = NC_{2,2,2}(6) \cup NC_{2,4}(6) \cup NC_6(6)$ .)

So, the formular (2.1) goes to

$$\sum_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = n} \sum_{\pi \in NC_{l_1, \dots, l_p}(n)} \left( p_0^{l_1} \dots p_0^{l_p} \right) \mu(\pi, 1_n). \blacksquare$$

**Example 3.1.** Let  $N = 2$ . Then  $A = L(F_2)$  and  $G = a + b + a^{-1} + b^{-1}$ , where  $F_2 = \langle a, b \rangle$ . Then

$$\begin{aligned} k_4(G, G, G, G) &= \sum_{\pi \in NC_{2,2}(4)} (p_0^2 p_0^2) \mu(\pi, 1_4) + p_0^4 \\ &= -2(p_0^2)^2 + p_0^4 = -32 + 28 \\ &= -4. \end{aligned}$$

and

$$\begin{aligned} k_6(G, \dots, G) &= \sum_{\pi \in NC_{2,2,2}(6)} (p_0^2 p_0^2 p_0^2) \mu(\pi, 1_6) \\ &\quad + \sum_{\pi \in NC_{2,4}(6)} (p_0^2 p_0^4) \mu(\pi, 1_6) + p_0^6 \\ &= (2(p_0^2)^3 + 2(p_0^2)^3 + (p_0^2)^3 + (p_0^2)^3 + (p_0^2)^3) \\ &\quad + |NC_{2,4}(6)| (p_0^2 p_0^4 (-1)) + p_0^6 \\ &= 448 - 672 + 232 = 8. \end{aligned}$$

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